

by Peter Winkler

Solution:

The state of locker n is changed when the k th student passes through, for every divisor k of n . Here, we make use of the fact that divisors *usually* come in pairs $\{j, k\}$ where $j \cdot k = n$ (including the pair $\{1, n\}$); so the net effect of students j and k on this locker is nil. The exception is when n is a perfect square, in which case there is no other divisor to cancel the effect of the \sqrt{n} th student; therefore, the lockers which are open at the end are exactly the perfect squares, 1, 4, 9, 16, 25, 36, 49, 64, 81, and 100. ♡

For the next puzzle you might like to know a certain handy rhyme, attributed to the mathematician Nathan Fine but inspired by a lovely proof by the great, late Paul Erdős:

Chebyshev said it and I say it again
There's always a prime between n and $2n$.

This fact is known as Bertrand's Postulate, proved by Chebyshev in 1852, with later (neater) proofs by Ramanujan and Erdős.

Factorial Coincidence

Suppose that a , b , c , and d are positive integers, all different, all greater than one. Can it be that $a!^b = c!^d$?

Solution:

Assume that such numbers exist, say with $a < c$, thus $b > d$. Then $c > 2$, and by Bertrand's Postulate, there is a prime p with $c/2 < p < c$. This p appears exactly d times in the prime factorization of $c!^d$ but either b times or not at all in the prime factorization of $a!^b$. The contradiction shows that the desired quartet (a, b, c, d) does not exist.

Even Split



Prove that from every set of $2n$ integers, you can choose a subset of size n whose sum is divisible by n .

Solution:

Call a set "flat" if it sums to 0 modulo n . Let us note first that the statement we want to prove implies the following seemingly weaker statement: If S is a flat set of $2n$ numbers, then S can be split into two flat sets of size n . However, that in turn implies that any set of only $2n - 1$ numbers contains a flat subset of size n because we can add a $2n$ th number to make the original set flat, then apply the previous statement to split this into *two* flat subsets of size n . One of these (the one without the new number) will do the trick.

So all three of these statements are equivalent. Suppose we can prove the second for $n = a$ and for $n = b$. Then if a set S of size $2n = 2ab$ sums to $0 \pmod{ab}$, it is, in particular, flat with respect to a , and we can peel off subsets S_1, \dots, S_{2b} of size a which are also flat with respect to a . Each of these subsets S_i has a sum we can write in the form ab_i . The numbers b_i now constitute a set of size $2b$ which sums to $0 \pmod{b}$, so we can split them into two sets of size b which are flat with respect to b . The unions of the sets S_i in each part are a bipartition of the original S into sets of size ab which are ab -flat, just what we wanted.

It follows that if we can prove the statement for $n = p$ prime, then we have it for all n . Let S be a set of size $2p$, with the idea of creating a p -flat subset of size p .

How can we create such a subset? One natural possibility is to pair up the elements of S and choose one element from each pair. Of course, if we do that, it will behoove us to ensure that the elements in each pair are different mod p , so our choice will not be of Hobson's variety. Can we do that?

Yes, order the elements of S modulo p (say, 0 through $p-1$) and consider the pairs (x_i, x_{i+p}) for $i = 1, 2, \dots, p$. If x_i were equivalent to $x_{i+p} \pmod{p}$ for some i , then $x_i, x_{i+1}, \dots, x_{i+p}$ would all be equivalent mod p and we could take p of them to make our desired subset.

Now that we have our pairs, we proceed by "dynamic programming." Let A_k be the set of all sums (mod p) obtainable by adding one number from each of the first k pairs. Then $|A_1| = 2$ and we claim $|A_{k+1}| \geq |A_k|$, and moreover, $|A_{k+1}| > |A_k|$ as long as $|A_k| \neq p$. This is because $A_{k+1} = (A_k + x_{k+1}) \cup (A_k + x_{k+1+p})$; thus if $|A_{k+1}| = |A_k|$, these two sets are identical, implying $A_k = A_k + (x_{k+1+p} - x_{k+1})$. This is impossible since p is prime and $x_{k+1+p} - x_{k+1} \not\equiv 0 \pmod{p}$, unless $|A_k| = 0$ or p .

Since there are p pairs, we must eventually have $|A_k| = p$ for some $k \leq p$, hence $|A_p| = p$ and, in particular, $0 \in A_p$. The statement of the puzzle follows. \heartsuit

Factorials and Squares

Consider the product $100! \cdot 99! \cdot 98! \cdot \dots \cdot 2! \cdot 1!$. Call each of the 100 factors $k!$ a "term." Can you remove one term and leave a perfect square?

Solution:

Perfect squares have some nice properties. For example, the product of any number of perfect squares is itself a perfect square: e.g., $A^2 \cdot B^2 \cdot C^2 = (A \cdot B \cdot C)^2$.

Let's call our big product N and observe that it's not far from already being a product of perfect squares. The product of N 's first two terms, for example, is $100 \cdot 99!^2$ (which happens to be a perfect square since $100 = 10^2$).

In fact, we could pair up all the terms and write N in the following form: $100 \cdot 99!^2 \cdot 98 \cdot 97!^2 \cdot 96 \cdot 95!^2 \cdot \dots \cdot 4 \cdot 3!^2 \cdot 2 \cdot 1!^2$, which is a perfect square